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Quantum group invariant supersymmetric *t–J* model with periodic boundary conditions

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Abstract. An integrable version of the supersymmetric *t–J* model which is quantum group invariant as well as periodic is introduced and analysed in detail. The model is solved through the algebraic nested Bethe ansatz method.

The Bethe ansatz method [1], first introduced to solve the XXX Heisenberg chain, is one of the most powerful tools in the treatment of integrable models. Its further development had important contributions from Yang and Yang [2] and Baxter [3], among others (for a review, see De Vega [4]). A great impetus in the theory of integrable systems was given by the quantum inverse scattering method [5]. This approach provides a unified framework for the exact solutions of classical and quantum models and led naturally to the new mathematical concept of quantum groups [6]. The construction of quantum group invariant integrable models has been attracting considerable attention. One possible way of obtaining such invariant models is to deal with open boundary conditions (OBC). In this connection, some quantum group invariant integrable models, such as the XXZ-Heisenberg model [7, 8], the $spl_q(2,1)$ supersymmetric t–J model [9, 10], the $SU_q(N)$ [11], the $SU_q(n/m)$ [12, 13] and the $B_n^{(1)}$, $C_n^{(1)}$ and $D_n^{(1)}$ spin chains [14] have been formulated. In particular, (with the exception of the $B_n^{(1)}$, $C_n^{(1)}$ and $D_n^{(1)}$ cases) their spectrum have been obtained through a generalization of the Sklyanin–Cherednik construction of the Yang–Baxter algebra [15, 16]. For these cases, the use of OBC resulted in the calculations becoming much more complex than for periodic boundary conditions (PBC). For instance, the commutation relations between the pseudoparticle operators $\mathcal{B}_\alpha$ and the transfer matrix are much more involved. In addition, the structure of the unwanted terms generated in the procedure is so complicated that only after sophisticated manipulations is it possible to recognize wanted and unwanted contributions. For the $B_n^{(1)}$, $C_n^{(1)}$ and $D_n^{(1)}$ chains, due to technical difficulties, a 'doubled' postulate has been proposed to obtain the spectrum.

Recently, the question as to whether quantum group invariance necessarily implies the use of OBC has been addressed in the literature. The construction of quantum group invariant integrable closed chains has been examined by some authors [17–19] and, in fact, a quantum group invariant XXZ model and an $U_q(sl(n))$ invariant chain with PBC have been formulated and analysed in detail. Therefore, it is of interest to find other quantum group invariant integrable closed chains.

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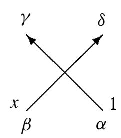
In this paper we introduce an integrable version of the supersymmetric t–J model which is quantum group invariant and periodic. The system is of interest because of its possible connection with high-T_c superconductivity. It describes electrons with nearest-neighbour hopping and spin exchange interaction on a chain (see equation (18)) and can be considered as an anisotropic extension of the supersymmetric t–J model. Its physical properties are, of course, essentially the same as for the case of OBC. Nevertheless, the approach adopted here drastically simplifies the nested Bethe ansatz. Moreover, this is the first time that a quantum supergroup invariant integrable periodic model has been presented. The corresponding Hamiltonian is related to a transfer matrix of a 'graded' vertex model [20] with anisotropy. The system is analysed through a generalization of the construction of [18] to the case of a 'graded' 15-vertex model and the Bethe ansatz equations are obtained.

We begin by introducing the R-matrix, which in terms of a generic spectral parameter x and a deformation parameter q reads [21]



$$R^{\gamma\delta}_{\alpha\beta}(x) = \; = \left(\begin{array}{ccc|ccc|ccc} a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & b & 0 & c_- & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & b & 0 & 0 & 0 & c_- & 0 & 0 \\ \hline 0 & c_+ & 0 & b & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & b & 0 & c_- & 0 \\ \hline 0 & 0 & c_+ & 0 & 0 & 0 & b & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & c_+ & 0 & b & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & w \end{array}\right) \tag{1}$$

where α, β (γ, δ) are column (row) indices running from 1 to 3 and

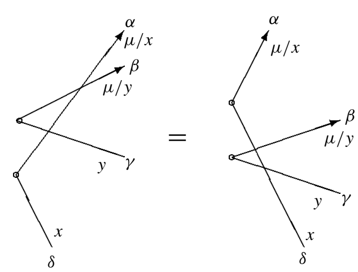
$$\begin{aligned} a &= xq - \frac{1}{xq} \qquad b = x - \frac{1}{x} \qquad c_+ = x\left(q - \frac{1}{q}\right) \\ c_- &= \frac{1}{x}\left(q - \frac{1}{q}\right) \qquad w = -\frac{x}{q} + \frac{q}{x}. \end{aligned} \tag{2}$$

The subscripts x and 1 in the diagram in equation (1) will soon become clear. The R-matrix (1) acts in the tensor product of two three-dimensional auxiliary spaces $\mathbb{C}^3 \otimes \mathbb{C}^3$ and it fulfills the Yang–Baxter equation

$$R^{\alpha''\beta''}_{\alpha'\beta'}(x/y) R^{\alpha'\gamma''}_{\alpha\ \gamma'}(x) R^{\beta'\gamma'}_{\beta\ \gamma}(y) = R^{\beta''\gamma''}_{\beta'\gamma'}(y) R^{\alpha''\gamma'}_{\alpha'\ \gamma}(x) R^{\alpha'\beta'}_{\alpha\ \beta}(x/y). \tag{3}$$

It is easy to check that it also satisfies the Cherdnik's reflection property [15]

$$R^{\alpha\ \beta}_{\alpha'\beta'}(x/y) R^{\beta'\alpha'}_{\gamma\ \delta}\left(\frac{\mu}{xy}\right) = R^{\alpha\ \beta}_{\alpha'\beta'}\left(\frac{\mu}{xy}\right) R^{\beta'\alpha'}_{\gamma\ \delta}(x/y)$$

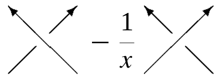


(4)

Here the symbol (○) indicates that at this point the spectral parameter changes from x to μ/x and y to μ/y. Note the presence of an arbitrary constant μ in the above equation, which is related with the choice of the boundaries. As in the case of the $U_q(sl(n))$ invariant integrable chain [18], we will take the limit $\mu \to \infty$ in order to construct a quantum group invariant model with PBC.

For later convenience the spectral parameter dependent R-matrix (1) can be written in terms of 'constant' R-matrices ($R_\pm$) as

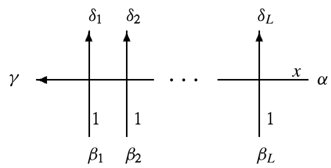
$$R(x) = xR_+ - \frac{1}{x}R_- = x \;\diagup\!\!\!\!\diagdown\; - \frac{1}{x} \;\diagup\!\!\!\!\diagdown \tag{5}$$



where R_+ (R_-) corresponds to the leading term in the limit of the matrix $R(x)$ for $x \to \infty(0)$.

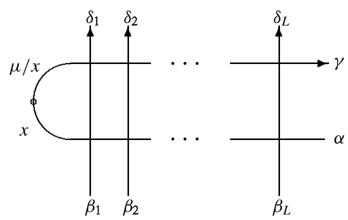
As usual, the standard monodromy matrix is defined as the product of R-matrices (1) as follows:

$$T^{\gamma\{\delta\}}_{\alpha\{\beta\}}(x) = R^{\gamma\ \delta_1}_{\alpha_1\beta_1}(1/x)R^{\alpha_1\delta_2}_{\alpha_2\beta_2}(1/x)\cdots R^{\alpha_{L-1}\delta_L}_{\alpha\ \beta_L}(1/x) = \tag{6}$$



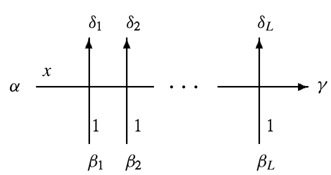
It acts in the tensor product of a L-dimensional 'quantum space' and a three-dimensional auxiliary space ($\mathbb{C}^{3L} \times \mathbb{C}^3$). For the case $q = 1$, taking the trace of the T-matrix (6) in the auxiliary space one gets an $spl(2,1)$ invariant transfer matrix, related with the supersymmetric t–J model [22]. However, for $q \neq 1$, this trace does not generate an $spl_q(2,1)$ invariant transfer matrix. Then, in order to construct a quantum group invariant integrable model we have to introduce the 'doubled' monodromy matrix $\mathcal{U}$:

$$\mathcal{U}^{\gamma\{\delta\}}_{\alpha\{\beta\}}(x,\{\mu\}) \;=\; \tilde{T}^{\gamma\{\delta\}}_{\alpha'\{\beta'\}}(\mu/x)\, T^{\alpha'\{\beta'\}}_{\alpha\{\beta\}}(x) = \tag{7}$$



where $\tilde{T}$ is a row-to-row monodromy matrix proportional to T^{-1}:

$$\tilde{T}^{\gamma\{\delta\}}_{\alpha\{\beta\}}(x) = R^{\delta_1\alpha_1}_{\beta_1\,\alpha}(x)R^{\delta_2\alpha_2}_{\beta_2\alpha_1}(x)\cdots R^{\delta_L\ \gamma}_{\beta_L\ \alpha_{L-1}}(x) = \tag{8}$$



and then take the appropriate trace in the auxiliary space. The arbitrary constant μ in (7) can be used to select the boundary conditions. Choosing $\mu = 1$, one obtains the $spl_q(2,1)$ invariant supersymmetric t–J model with open boundary conditions (OBC), already discussed in [9, 10]. Other quantum group invariant integrable models, such as the XXZ model [7, 8], the $SU_q(N)$ [11] and $SU_q(n/m)$ [12, 13] chains have also been considered in connection with OBC.

In what follows, we consider the limit $\mu \to \infty$. In this limit the contributions from $\tilde{T}$ to the monodromy matrix $\mathcal{U}$ and consequently to the transfer matrix (see equation (11)) reduce to a product of constant R-matrices (R_+) (see equation (5)). We will prove that this choice yields a quantum group invariant supersymmetric t–J model with PBC.

The 'doubled' monodromy matrix $\mathcal{U}$ (7) can be represented as a 3×3 matrix whose entries are matrices acting on the 'quantum space'

$$\mathcal{U}_{\alpha}^{\gamma}(x) = \left(\begin{array}{c|cc} \mathcal{A}(x) & \mathcal{B}_2(x) & \mathcal{B}_3(x) \\ \hline \mathcal{C}_2(x) & \mathcal{D}_2^2(x) & \mathcal{D}_3^2(x) \\ \mathcal{C}_3(x) & \mathcal{D}_2^3(x) & \mathcal{D}_3^3(x) \end{array} \right). \tag{9}$$

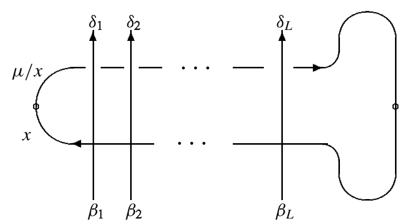
Using equations (3) and (4) (already in the limit $\mu \to \infty$) we can prove that it fulfills the following Yang–Baxter relation:

$$R_{\alpha'\beta'}^{\alpha\ \beta}(y/x)\mathcal{U}_{\gamma'}^{\beta'}(x)R_{+\gamma\ \delta'}^{\gamma'\alpha'}\mathcal{U}_{\delta}^{\delta'}(y) = \mathcal{U}_{\alpha'}^{\alpha}(y)R_{+\delta'\beta'}^{\alpha'\ \beta}\mathcal{U}_{\gamma'}^{\beta'}(x)R_{\gamma\ \delta}^{\gamma'\delta'}(y/x). \tag{10}$$

We observe in the above equation the presence of constant R-matrices (R_+) instead of spectral parameter dependent R-matrices, which appear in the corresponding relation using OBC [9, 10]. This will simplify the algebraic nested Bethe ansatz considerably.

Finally, the transfer matrix is defined as the Markov trace associated with the superalgebra $spl_q(2,1)$ (K_{α}^{α}) of the 'doubled' monodromy matrix in the auxiliary space:

$$\mathcal{T}_{\{\beta\}}^{\{\delta\}}(x) = \sum_{\alpha} K_{\alpha}^{\alpha}\mathcal{U}_{\alpha\{\beta\}}^{\alpha\{\delta\}} = \tag{11}$$



where

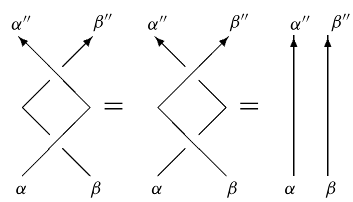
$$K_{\alpha}^{\alpha} = \sigma_{\alpha}\, q^{\left(-2\sum_{\gamma}^{\alpha-1}\sigma_{\gamma}\right)-\sigma_{\alpha}+1} \tag{12}$$

and

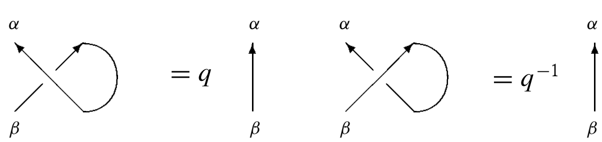
$$\sigma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{13}$$

The Yang–Baxter equation for the 'doubled' monodromy matrix $\mathcal{U}$ (10) implies that the transfer matrix (11) commutes for different spectral parameters, which proves the integrability of the model. Then, from the above defined transfer matrix and the following properties:

$$R_{\pm\alpha'\beta'}^{\ \alpha''\beta''}R_{\mp\beta\alpha}^{\ \beta'\alpha'} = \delta_{\alpha}^{\alpha''}\delta_{\beta}^{\beta''} \tag{14}$$



$$R_{\pm\alpha'\beta}^{\ \alpha\alpha'}K_{\alpha'}^{\alpha'} = q^{\pm 1}\delta_{\beta}^{\alpha} \tag{15}$$



we obtain a quantum group invariant one-dimensional supersymmetric t–J model with PBC through

$$\mathcal{H} \propto \frac{\partial}{\partial x}\ \ln(\mathcal{T})|_{x=1}. \tag{16}$$

This yields

$$\mathcal{H} = \sum_{j=1}^{L-1} h_j + h_0 \tag{17}$$

where

$$h_j = -\sum_\sigma (c^\dagger_{j,\sigma} c_{j+1,\sigma} + c^\dagger_{j+1,\sigma} c_{j,\sigma}) - \cos\gamma\, n_j + 2\cos\gamma$$
$$-2\left[S^x_j S^x_{j+1} + S^y_j S^y_{j+1} + \cos\gamma\left(S^z_j S^z_{j+1} - \frac{n_j n_{j+1}}{4}\right)\right]$$
$$+i\sin(\gamma)(n_j - n_{j+1}) - i\sin(\gamma)(n_j S^z_{j+1} - S^z_j n_{j+1}) \tag{18}$$

and h_0 is a boundary term given by

$$h_0 = \underbrace{\hat{R}^-_1 \hat{R}^-_2 \cdots \hat{R}^-_{L-1}}_{G}\, h_{L-1}\, \underbrace{\hat{R}^+_{L-1} \cdots \hat{R}^+_2 \hat{R}^+_1}_{G^{-1}} \tag{19}$$

with

$$\hat{R}^{\pm\,\{\gamma\}}_{j\;\{\beta\}} = \mathbb{1}^{\gamma_1}_{\beta_1} \otimes \mathbb{1}^{\gamma_2}_{\beta_2} \otimes \cdots R^{\pm\,\gamma_j\gamma_{j+1}}_{\;\;\beta_{j+1}\beta_j} \otimes \cdots \mathbb{1}^{\gamma_L}_{\beta_L} \qquad j = 1, 2, \ldots, L-1. \tag{20}$$

The presence of this boundary term (h_0) is essential for the construction of a quantum group invariant model with PBC. Note that it emerges naturally from the present construction. The other possible way of obtaining a quantum group invariant Hamiltonian ($h_0 = 0$, which corresponds to OBC), was already discussed in [9, 10]. In equation (18) L is the number of sites of the quantum chain, the $c^{(\dagger)}_{j\pm}$ are spin-up or -down annihilation (creation) operators, the $\boldsymbol{S}_j$ are spin matrices and the n_j are occupation numbers of electrons at lattice site j. The operators H, h_i and $\hat{R}^\pm_i$ ($i = 1, 2, \ldots, L-1$) act on the 'quantum space' $\mathbb{C}^{3L}$ (for simplicity, we omit the quantum space indices and write them only whenever necessary).

It was shown in [18] using methods of topological quantum field theory that the transfer matrix obtained by this approach for an $U_q(sl(n))$ invariant chain is equivalent to the partition function of a vertex model on a torus and the periodicity of that model is evident from this. However, here it is not obvious that the Hamiltonian (17) describes a model with PBC. To prove this fact we first note that the operators $\hat{R}^\pm$ are a representation of the Hecke algebra [23]†

$$\begin{aligned}
&\hat{R}^\pm_j \hat{R}^\pm_j = \pm(q - 1/q)\hat{R}^\pm_j + \mathbb{1}\\
&\hat{R}^\pm_j \hat{R}^\pm_{j\pm1} \hat{R}^\pm_j = \hat{R}^\pm_{j\pm1} \hat{R}^\pm_j \hat{R}^\pm_{j\pm1}\\
&\hat{R}^\pm_i \hat{R}^\pm_j = \hat{R}^\pm_j \hat{R}^\pm_i \qquad |i-j| \geqslant 2.
\end{aligned} \tag{21}$$

From the Hecke algebra conditions (21) and the following relation:

$$h_j = -\hat{R}^\pm_j + q^{\pm1}\mathbb{1} \qquad j = 1, 2, \ldots, L-1 \tag{22}$$

we find that the operator G^{-1} maps h_j into h_{j-1}

$$G^{-1} h_j G \,=\, h_{j-1} \qquad j = 2, \ldots, L-1 \tag{23}$$

and h_1 into h_0

$$G^{-1} h_1 G = G h_{L-1} G^{-1}. \tag{24}$$

† To obtain relations (21), the Yang–Baxter algebra (3) and equation (20) have been used.

Then, denoting the Hamiltonian of equation (17) by $\mathcal{H}_{1,2,\ldots,L}$ and that obtained by cyclic permutation $(1, 2, \ldots, L) \to (L, 1, 2, \ldots, L-1)$ by $\mathcal{H}_{L,1,2,\ldots,L-1}$, and using the properties (23), (24), we show that

$$\mathcal{H}_{L,1,2,\ldots,L-1} = G^{-1}\mathcal{H}_{1,2,\ldots,L}G \tag{25}$$

i.e. both Hamiltonians are physically equivalent, which completes the proof that we are dealing with a periodic chain.

Note that, although the boundary term (19) is apparently non-local, it is local in the sense that it commutes with the local observables, in particular, the generators of the Hecke algebra [24]

$$[h_0\,,\ \hat{R}_j^{\pm}] = 0 \qquad 1 < j < L-1. \tag{26}$$

This can be verified by using equations (21) and (22). Finally, the quantum group invariance of the Hamiltonian (17) follows directly from the fact that the operators $\hat{R}^{\pm}$ are a representation of the Hecke algebra.

Next we solve the eigenvalue problem of the transfer matrix

$$\mathcal{T}\Psi = (\mathcal{A} + q^{-2}\mathcal{D}_2^2 - q^{-2}\mathcal{D}_3^3)\Psi = \Lambda\Psi \tag{27}$$

(and consequently that of the Hamiltonian (17)) through the algebraic nested Bethe ansatz (ANBA) with two levels. According to the first-level Bethe ansatz, the vector Ψ can be written as

$$\Psi = \sum_{\{\alpha\}=2}^{3} \mathcal{B}_{\alpha_1}(x_1)\mathcal{B}_{\alpha_2}(x_2)\cdots\mathcal{B}_{\alpha_r}(x_r)\Psi_{(1)}^{\{\alpha\}}\Phi. \tag{28}$$

The coefficients $\Psi_{(1)}$ are determined later by the second-level Bethe ansatz while Φ is the reference state defined by the equation

$$\mathcal{C}_\alpha\Phi = 0 \qquad \text{for} \quad \alpha = 2, 3$$

whose solution is $\Phi = \otimes_{i=1}^{L}|1\rangle_i$. It is an eigenstate of $\mathcal{A}$ and $\mathcal{D}_\beta^\alpha$:

$$\mathcal{A}(x)\Phi = q^L\, a(1/x)^L\Phi \tag{29}$$

$$\mathcal{D}_\beta^\alpha(x)\Phi = \delta_\beta^\alpha b(1/x)^L\Phi. \tag{30}$$

Following the general strategy of the algebraic nested Bethe ansatz we apply the transfer matrix (11) to the eigenvector Ψ (28). Using the following commutation rules derived from the Yang–Baxter relations (10):

$$\mathcal{A}(x)\mathcal{B}_\alpha(y) = \frac{1}{q}\frac{a(x/y)}{b(x/y)}\mathcal{B}_\alpha(y)\mathcal{A}(x) - \frac{1}{q}\frac{c_-(x/y)}{b(x/y)}\mathcal{B}_\alpha(x)\mathcal{A}(y)$$
$$-\frac{q-1/q}{q}\sum_{\beta=2}^{3}\mathcal{B}_\beta(x)\mathcal{D}_\alpha^\beta(y) \tag{31}$$

$$\mathcal{D}_\beta^\gamma(x)\mathcal{B}_\alpha(y) = R_{+\ \gamma'\delta'}^{\ \alpha'\gamma}\frac{R_{\beta\ \alpha}^{\beta'\gamma'}(y/x)}{b(y/x)}\mathcal{B}_{\alpha'}(y)\mathcal{D}_{\beta'}^{\delta'}(x) - R_{+\ \beta\beta'}^{\ \alpha'\gamma}\frac{c_-(y/x)}{b(y/x)}\mathcal{B}_{\alpha'}(x)\mathcal{D}_\alpha^{\beta'}(x) \tag{32}$$

we commute $\mathcal{A}$ and $\mathcal{D}$ with all $\mathcal{B}$'s and apply them to the reference state Φ. All indices in equations (31) and (32) assume only the values 2, 3. We begin by considering the action of $\mathcal{A}$ on Ψ. Using equation (31), two types of terms arise when $\mathcal{A}$ passes through $\mathcal{B}_\alpha$. In the first $\mathcal{A}$ and $\mathcal{B}_\alpha$ preserve their arguments, and in the second their arguments are exchanged. The first kind of terms are called 'wanted terms', since they can originate a vector proportional to Ψ; this cannot happen to the second type and therefore they are called the 'unwanted terms' (UT). Note that in the present formulation ($\mu \to \infty$) the decomposition into wanted and unwanted terms appears naturally, as in the usual periodic case (where the transfer matrix, which is not quantum group invariant, is constructed by taking the trace of the standard row-to-row monodromy matrix). This is in contrast to the case of OBC ($\mu = 1$), where it is necessary to redefine the $\mathcal{D}$-operators in order to recognize wanted and unwanted contributions [9, 10]. After successive applications of (31), together with (29), we obtain

$$\mathcal{A}(x)\Psi = q^{L-r} a(1/x)^L \prod_{i=1}^{r} \frac{a(x/x_i)}{b(x/x_i)} \Psi + \text{UT}. \tag{33}$$

Correspondingly from the commutation relations between $\mathcal{D}$ and $\mathcal{B}$, (32) and the action of $\mathcal{D}$ on the reference state Φ (30), we obtain wanted and unwanted contributions:

$$q^{-2}(\mathcal{D}_2^2 - \mathcal{D}_3^3)\Psi = b(1/x)^L \prod_{i=1}^{r} \frac{1}{b(x_i/x)} \sum_{\{\alpha'\}=2}^{3} \mathcal{B}_{\alpha'_1}(x_1)\mathcal{B}_{\alpha'_2}(x_2)\cdots\mathcal{B}_{\alpha'_r}(x_r) q^{-1} \mathcal{T}_{(1)}^{\{\alpha'\}} \Psi_{(1)} + \text{UT}. \tag{34}$$

Here we have introduced a new (second-level) transfer matrix

$$\mathcal{T}_{(1)} = \sum_{\alpha=2}^{3} \sigma_\alpha \, q^{-1} \mathcal{U}_{(1)}{}_{\alpha}^{\alpha} \tag{35}$$

as the Markov trace associated with the superalgebra $SU_q(1,1)$ of the second level 'doubled' monodromy matrix $\mathcal{U}_{(1)}$, defined analogously to $\mathcal{U}$ (see equation (9)). Now, all indices range from 2 to 3, as in the internal block of the matrix $\mathcal{U}$ (9). Thus, we will treat the internal block $\mathcal{D}$ in the same way as we have done with the whole matrix, through the identification $\mathcal{A}_{(1)} \equiv \mathcal{U}_{(1)}{}_2^2$, $\mathcal{B}_{(1)} \equiv \mathcal{U}_{(1)}{}_3^2$, $\mathcal{C}_{(1)} \equiv \mathcal{U}_{(1)}{}_2^3$ and $\mathcal{D}_{(1)} \equiv \mathcal{U}_{(1)}{}_3^3$, The first term (wanted term) on the right-hand side of (34) is proportional to Ψ if the eigenvalue equation

$$\mathcal{T}_{(1)}\Psi_{(1)} = \Lambda_{(1)}\Psi_{(1)} \tag{36}$$

is satisfied. The eigenvector $\Psi_{(1)}$ of $\mathcal{T}_{(1)}$ is defined by the second-level Bethe ansatz

$$\Psi_{(1)} = \mathcal{B}_{(1)}(y_1)\mathcal{B}_{(1)}(y_2)\cdots\mathcal{B}_{(1)}(y_m)\Phi_{(1)} \tag{37}$$

where $\Phi_{(1)}$ is the second level reference state given by $\Phi_{(1)} = \otimes_{i=1}^{r} |2\rangle_i$, as a result of being annihilated by $\mathcal{C}_{(1)}$. Then, proceeding along the same lines as in the previous step, we apply $\mathcal{T}_{(1)}$, equation (35), to the state $\Psi_{(1)}$, equation (37), and pass $\mathcal{A}_{(1)}$ and $\mathcal{D}_{(1)}$ through the $\mathcal{B}_{(1)}$'s, using commutation relations derived from the Yang–Baxter relation (10) and the action of $\mathcal{A}_{(1)}$ and $\mathcal{D}_{(1)}$ on the vacuum $\Phi_{(1)}$. As before, we obtain wanted and unwanted contributions:

$$\mathcal{A}_{(1)}(x)\Psi_{(1)} = q^{-m+r} \prod_{i=1}^{r} a(x_i/x) \prod_{j=1}^{m} \frac{a(x/y_j)}{b(x/y_j)} \Psi_{(1)} + \text{UT} \tag{38}$$

$$\mathcal{D}_{(1)}(x)\Psi_{(1)} = (-1)^m q^{-m} \prod_{i=1}^{r} b(x_i/x) \prod_{j=1}^{m} \frac{w(y_j/x)}{b(y_j/x)} \Psi_{(1)} + \text{UT}. \tag{39}$$

Then, combining equations (38), (39), (34), (33) and (27) we get the eigenvalue $\Lambda(x)$ of the transfer matrix $\mathcal{T}$ if the ' unwanted terms' cancel out:

$$\Lambda(x) = q^{L-r} a(1/x)^L \prod_{i=1}^{r} \frac{a(x/x_i)}{b(x/x_i)} + q^{-2+r-m} b(1/x)^L \prod_{i=1}^{r} \frac{a(x_i/x)}{b(x_i/x)} \prod_{j=1}^{m} \frac{a(x/y_j)}{b(x/y_j)}$$

$$-(-1)^m q^{-2-m} b(1/x)^L \prod_{j=1}^{m} \frac{w(y_j/x)}{b(y_j/x)}. \qquad (40)$$

All unwanted terms vanish if the Bethe ansatz equations hold. They can be obtained by demanding that the eigenvalue $\Lambda(x)$ (40) has no poles at $x = x_i$ and $x = y_j$, since $\mathcal{T}$ is an analytic function in x

$$q^{L+m+2-2r} \left(\frac{a(1/x_k)}{b(1/x_k)} \right)^L \prod_{i=1}^{r} \frac{a(x_k/x_i)}{b(x_k/x_i)} \frac{b(x_i/x_k)}{a(x_i/x_k)} \prod_{j=1}^{m} \frac{b(x_k/y_j)}{a(x_k/y_j)} = -1 \qquad k = 1, \ldots r \qquad (41)$$

$$(-1)^m q^r \prod_{i=1}^{r} \frac{a(x_i/y_l)}{b(x_i/y_l)} \prod_{j=1}^{m} \frac{a(y_l/y_j)}{b(y_l/y_j)} \frac{b(y_j/y_l)}{w(y_j/y_l)} = 1 \qquad l = 1, \ldots m. \qquad (42)$$

Therefore, we have reduced the eigenvalue problem of the transfer matrix $\mathcal{T}$ to a system of coupled algebraic equations in the parameters x and y. Note that these equations are much simpler than those obtained for OBC (see [9, 10]). A possible application of these results (equations (41), (42)) would be an analysis of the structure of the ground state and some low lying excitations of the model in the thermodynamic limit. The question of the completeness of the Bethe states for an $spl_q(2,1)$ invariant supersymmetric t–J model is left open. In [22], this point was treated for the isotropic case ($q = 1$), where a complete set of eigenvectors was obtained by combining the Bethe ansatz with the $spl(2,1)$ underlying supersymmetry of the model. The completeness of its q-deformed version is under investigation.

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